Nonexpansive Maps and the Horofunction Boundary

Brian Lins

Hampden-Sydney College

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Nonexpansive Maps

Definition

Let (M, d) be any metric space. We say that a map $f : M \to M$ is **nonexpansive** if

$$d(f(x), f(y)) \leq d(x, y)$$

for all $x, y \in M$.

In a Banach space, a nonexpansive map is one such that

$$||f(x) - f(y)|| \leq ||x - y||$$

for all x and y.

Basic Properties

Lemma

Let $f : M \to M$ be a nonexpansive map on a metric space M. Then the limit

$$A = \lim_{k \to \infty} \frac{d(f^k(x), x)}{k}$$

exists and does not depend on x.

We will refer to A as the **linear rate of growth** of f.

If a nonexpansive map has a fixed point, then the linear rate of growth is zero. *The converse is not true, however!*

Nonexpansive Maps in a Banach Space

If our metric space is a Banach space, then we can say even more.

Theorem (Kohlberg and Neyman)

Let C be a convex subset of a normed space X and let $f : C \to C$ be nonexpansive. Then there exists a linear functional $\varphi \in X^*$ with $||\varphi|| = 1$ such that for every $x \in C$,

$$\lim_{k \to \infty} \varphi\left(\frac{f^k(x)}{k}\right) = \lim_{k \to \infty} \left|\left|\frac{f^k(x)}{k}\right|\right| = \inf_{y \in C} ||f(y) - y||.$$

The Horofunction Boundary

Let (M, d) be a proper metric space. We can embed M into $C(M)$, the continuous real-valued functions on M by the mapping

$$\Phi : M \to C(M) \quad \text{defined} \quad \Phi(x) = d(x, \cdot) - d(x, z)$$

where z is a fixed reference point. Assuming that M is not compact, $\Phi(M)$ will not be closed in $C(M)$. The closure of $\Phi(M)$ is called the **Busemann compactification** of M. The **horofunction boundary** of M is the boundary of the Busemann compactification. We denote it $M(\infty)$

$$M(\infty) = \operatorname{cl}\Phi(M) \backslash \Phi(M).$$

The Horofunction Boundary

If X is a Banach space, a horofunction has the form:

$$h(y) = \lim_{k \to \infty} ||y - x_k|| - ||x_k||$$

where x_k is a sequence of points in X such that the limit converges uniformly on all compact subsets of X.

Properties of Horofunctions

Suppose that $h(y) = \lim_{k \to \infty} ||y - x_k|| - ||x_k||$ is a horofunction defined for all y in a Banach space X. The following properties follow immediately

- $h(y)$ is Lipschitz continuous with Lipschitz constant 1.
- $h(y)$ is convex.

Intuitively, horofunctions represent distances to a point at infinity.

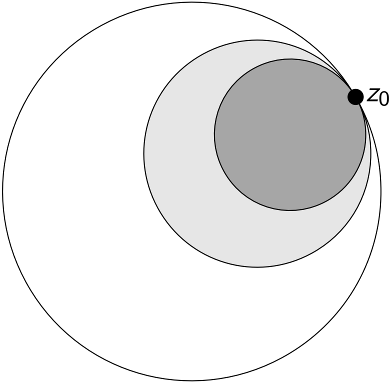
A Little Background

The horofunction boundary has been most widely studied in hyperbolic geometry and in several complex variables. Its history can be traced back to Wolff's theorem from complex analysis.

Theorem (Wolff's theorem)

Suppose that f is a holomorphic map from the open unit disc D into itself. Either f has a fixed point in D, or there exists a point z_0 on the boundary of the unit disc such that every open disc in D that is internally tangent to z_0 is invariant under f.

Wolff's Theorem



Horofunctions and Nonexpansive Maps

Beardon generalized Wolff's theorem to the following result.

Theorem (Beardon)

Let X be any proper CAT(0) space. Suppose that $f : X \to X$ is nonexpansive and has no fixed point in X. Then there exists a horofunction h such that $h(f(x)) \leq h(x)$ for all $x \in X$. In particular, the sublevel sets of h, called **horoballs**, *are invariant.*

This implies Wolff's theorem once you know two facts:

1. **The Schwarz-Pick lemma:** Holomorphic self-maps of the unit disc are nonexpansive with respect to the Poincare metric.
2. Horoballs in the Poincare metric are actually discs.

Finite Dimensional Normed Spaces

In a finite dimensional normed space, we can say even more about the interaction between a nonexpansive map and the horofunction boundary.

Theorem (Main Result)

Let C be a closed convex subset of a finite dimensional real Banach space X. Let $f : C \to C$ be nonexpansive and suppose that f has no fixed point in C. Then there is a horofunction h such that $\lim_{k\to\infty} h(f^k(x)) = -\infty$ for all $x \in C$.

Finite Dimensional Normed Spaces

We need two lemmas in order to prove the main result.

Lemma (No Fixed Point $\rightarrow$ Unbounded Orbit)

Suppose that C is a closed convex subset of X and $f : C \rightarrow C$ is nonexpansive. If f has no fixed point in C, then for every $x \in C$ the orbit $\{f^k(x)\}$ is not bounded.

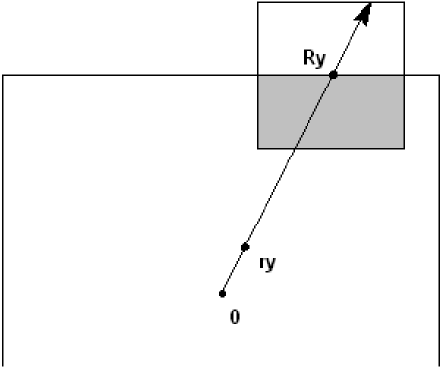
Proof.

Dafermos and Slemrod have shown that f is an invertible isometry on the accumulation points of any of its orbits. Therefore, by taking the intersection of a family of large closed balls centered at each accumulation point, we get a closed, bounded, convex subset of X that is invariant under f. This gives a contradiction by the Brouwer fixed point theorem. $\square$

Finite Dimensional Normed Spaces

Lemma

Let $y \in X$ be an element with $||y|| = 1$. Let $0 < \lambda < 1$. For any $R > r > 0$ and any $z \in X$ with $||z|| \leq R$, if $||z - Ry|| \leq \lambda R$, then $||z - ry|| \leq R - (1 - \lambda)r$.



Proof of Lemma

Proof.
By scaling $||z - Ry|| \leq \lambda R$ we get:

$$||(r/R)z - ry|| \leq \lambda r.$$

By the triangle inequality,

$$||z - ry|| \leq ||z - (r/R)z|| + ||(r/R)z - ry|| \leq$$

$$(1 - r/R)||z|| + \lambda r \leq R - r + \lambda r = R - (1 - \lambda)r.$$

$\square$

Proof of main result.

Fix an $x_0 \in C$ and let $x_k = f^k(x_0)$ for $k \geq 1$. The orbit x_k is not bounded since there is no fixed point. Therefore, we may choose an increasing sequence of integers k_i such that $\lim_{k_i \to \infty} ||x_{k_i}|| = \infty$ and for each k_i,

$$||x_{k_i}|| > ||x_m|| \text{ for all } m < k_i. \tag{1}$$

We may also assume that we have choosen the k_i sparse enough so that the vectors x_{k_i} converge in direction, that is

$$\lim_{k_i \to \infty} \frac{x_{k_i}}{||x_{k_i}||} = y.$$

We may choose k_i so that the convergence above is very rapid, say

$$\left|\left| \frac{x_{k_i}}{||x_{k_i}||} - y \right|\right| < 2^{-i}.$$

Given such a sequence k_i, we claim that

$$||x_{k_i} - x_{k_j - m}|| \leq ||x_{k_j}|| - \frac{1}{4}||x_{k_i}|| \tag{2}$$

whenever i and m are fixed, $m \geq 0$ and j is sufficiently large. Assume the equation above for now. The Ascoli-Arzela theorem allows us to construct the horofunction

$$h(x) = \lim_{j \to \infty} ||x - x_{k_j}|| - ||x_{k_j}||$$

by taking a subsequence of k_j.

By the nonexpansiveness of f, we observe that

$$h(x_{k_i+m}) = \lim_{j\to\infty} ||x_{k_i+m} - x_{k_j}|| - ||x_{k_j}|| \leq$$

$$\leq \liminf_{j\to\infty} ||x_{k_i} - x_{k_j-m}|| - ||x_{k_j}||.$$

Now, by eq. (2), $||x_{k_i} - x_{k_j-m}|| \leq ||x_{k_j}|| - \frac{1}{4}||x_{k_i}||$ for j sufficiently large. Therefore we obtain

$$h(x_{k_i+m}) \leq -\frac{1}{4}||x_{k_i}|| \quad \text{for all } m > 0.$$

Since

$$h(x_{k_i+m}) \leq -\frac{1}{4}||x_{k_i}|| \quad \text{for all } m > 0,$$

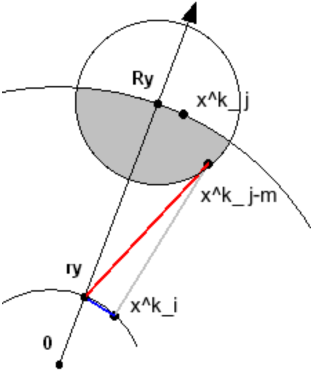
it follows that $h(x_k) \to -\infty$. In otherwords, the orbit x_k gets deeper and deeper into the horofunction. Since f is nonexpansive and horofunctions are Lipschitz of order one, it follows that

$$\lim_{k \to \infty} h(f^k(x)) = -\infty$$

for all $x \in C$.

Fix some $i \geq 1$ and $m \geq 0$. It remains to prove eq. (2):

$$||x_{k_i} - x_{k_{j-m}}|| \leq ||x_{k_j}|| - \frac{1}{4}||x_{k_i}|| \quad \text{for large } j.$$



Let $R = ||x_{k_j}||$ and $r = ||x_{k_i}||$.

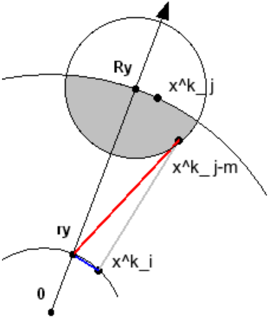
In order to use the lemma to estimate $||x_{k_j-m} - ry||$, we need to show that x_{k_j-m} is close to Ry.

Note that $||x_{k_j-m} - Ry||$ can be estimated by

1. $||x_{k_j-m} - x_{k_j}|| \leq ||x_0 - x_m|| = C(m)$
2. $||x_{k_j} - Ry|| \leq 2^{-j}R$

Since $R = ||x_{k_j}||$ grows infinitely large as $j \to \infty$, we may assume that $||x_{k_j-m} - Ry|| \leq \frac{1}{4}R$.

We can now use the lemma from before with $\lambda = \frac{1}{4}$ and we get $||x_{k_j - m} - ry|| \leq R - \frac{3}{4} r = ||x_{k_j}|| - \frac{3}{4} ||x_{k_i}||$.



We also have $||x_{k_i} - ry|| < 2^{-i} ||x_{k_i}||$.

Putting it all together, we get

$$||x_{k_{j}-m} - x_{k_i}|| \leq ||x_{k_j}|| - \frac{3}{4}||x_{k_i}|| + 2^{-i}||x_{k_i}||,$$

thus $||x_{k_{j}-m} - x_{k_i}|| \leq ||x_{k_j}|| - \frac{1}{4}||x_{k_i}||$ for $i \geq 1$. $\square$

A Nice Consequence

Theorem

Let C be a closed convex set in a finite dimensional real Banach space X, and suppose that $f : C \to C$ is nonexpansive. If f has no fixed point in C, then there is a linear functional $\varphi \in X^*$ such that $\lim_{k\to\infty} \varphi(f^k(x)) = \infty$ for all $x \in C$.

Proof.

There is a horofunction h such that $h(f^k(x)) \to -\infty$. Since horofunctions are convex, there is a subgradient $-\varphi \in X^*$ and an $x_0 \in X$ such that $h(x) - h(x_0) \geq -\varphi(x - x_0)$ for all $x \in X$. It follows that $\varphi(f^k(x)) \geq -h(f^k(x)) + C$ so $\varphi(f^k(x)) \to \infty$. $\square$

Other Settings

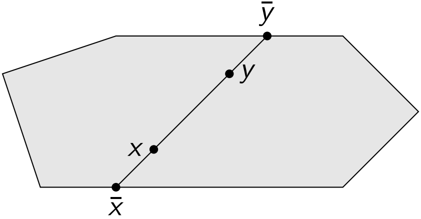
- The main theorem cannot be extended to infinite dimensional Banach spaces. Edelstein constructed two well known examples of nonexpansive maps in infinite dimensions that have bounded orbits but no fixed points.
- The main theorem also cannot be extended to CAT(0) spaces. There are is a simple counterexample which is a nonexpansive map in the Poincare metric on the unit disc.

$$f(z) = \frac{(1-2i)z-1}{z-(1+2i)}$$

Hilbert Geometry

Let Σ be any open convex subset of $\mathbb{R}^n$. The Hilbert metric on Σ is defined as

$$d_H(x, y) = \log \left(\frac{||x - \bar{y}|| \, ||y - \bar{x}||}{||x - \bar{x}|| \, ||y - \bar{y}||} \right).$$



Hilbert Geometry

Hilbert geometry has the following nice properties.

- Straight lines are geodesics.
- Balls are convex.
- The open unit disk in $\mathbb{C}$ with d_H is the Klein disk model for a hyperbolic space.
- If Σ is a polyhedral domain, then the Hilbert geometry on Σ is "almost isometric" to a finite dimensional Banach space.

A Denjoy-Wolff Theorem for Hilbert Geometry

Theorem

If Σ is a polyhedral domain and $f : \Sigma \to \Sigma$ is a Hilbert metric nonexpansive map with no fixed point in Σ, then the accumulation points of any orbit of f are all contained in a convex subset of the boundary of Σ.

A Denjoy-Wolff Theorem for Hilbert Geometry

Proof.

Since (Σ, d_H) is almost isometric to a finite dimensional normed space, there is a horofunction h on Σ such that $\lim_{k\to\infty} h(f^k(x)) = -\infty$ for all $x \in \Sigma$. Furthermore, h is a convex function. All accumulation points of $\{f^k(x)\}$ will therefore be contained in the following closed convex subset of the boundary:

$$\bigcap_{M<0} \mathrm{cl}\,\{y \in \Sigma : h(y) < M\}.$$

$\square$